Factorial notation

The definition of $n!$ is as follows:

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 1$$

where n is a non-negative integer.

The factorial function satisfies the relationship $n! = n \times (n-1)!$. If we put $n = 1$, this tells us that $1! = 1 \times 0! \Rightarrow 0! = 1$.

Question 1.1

Evaluate 5!

Example

Simplify the expression $\dfrac{n!(2n-1)!}{(2n)!(n-2)!}$.

Solution

This expression can be written as:

$$\frac{n!}{(n-2)!} \times \frac{(2n-1)!}{(2n)!}$$

The first factor equals $n(n-1)$, because $n!$ contains an extra n and $n-1$ in its expansion, which are not contained in the expansion of $(n-2)!$, and the second factor is $\dfrac{1}{2n}$.

So we get $\dfrac{n!(2n-1)!}{(2n)!(n-2)!} = \dfrac{n(n-1)}{2n} = \tfrac{1}{2}(n-1)$.

Question 1.2

Simplify the expression $\dfrac{(3n)!(2n+1)!}{(3n+2)!(2n-1)!}$.

Gamma function

$\Gamma(x)$, where $x > 0$, is defined by the integral $\int_0^\infty t^{x-1} e^{-t}\, dt$. This is the gamma function.

This function is used in statistics in connection with the gamma distribution, and there are several properties of the function that you are going to need to know. Here we will quote the results without proof.

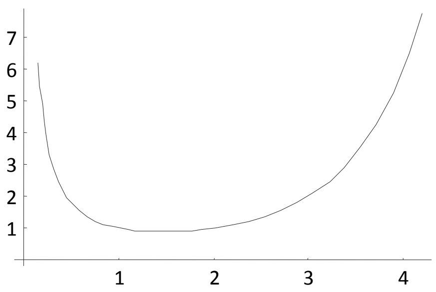
Result 1: $\Gamma(x) = (x-1)\Gamma(x-1)$ where $x > 1$

Result 2: $\Gamma(n) = (n-1)!$ where $n = 1, 2, 3, \ldots$

Result 3: $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

These results can be found on page 5 of the *Tables*.

The graph of the gamma function looks like this:



Example

Evaluate $\Gamma(6.5)$.

Solution

Using Result 1:

$$\Gamma(6.5) = 5.5\Gamma(5.5)$$

By repeating this process:

$$\begin{aligned}\Gamma(6.5) &= 5.5 \times 4.5 \times 3.5 \times 2.5 \times 1.5 \times 0.5 \times \Gamma(0.5) \\ &= 162.42\sqrt{\pi} \\ &= 287.89\end{aligned}$$

Question 1.3

Evaluate $\Gamma(5)$.

Question 1.4

Evaluate $\Gamma(4.5)$.

Question 1.5

Simplify $\dfrac{\Gamma(n-1)(2n-1)!}{\Gamma(2n)(n+1)!}$ where *n* is a natural number.

Question 1.6

Show that, if n is a whole number, $\Gamma(n + ½) = \dfrac{(2n)!}{2^{2n}\, n!}\sqrt{\pi}$.

Stirling's approximation

Working out a factorial by multiplying all the numbers together would be very tedious for large values of n. The following approximations (called Stirling's approximation) are sometimes useful in derivations involving large values of n:

$$n! \approx n^{n+½} e^{-n} \sqrt{2\pi}$$

and $$\Gamma(n) \approx n^{n-½} e^{-n} \sqrt{2\pi}$$

(Don't worry that these appear to be inconsistent with the relationship $\Gamma(n) = (n-1)!$ Remember that for very large values of n, the ratio $\dfrac{n-1}{n}$ is very close to 1.)

Solutions

Solution 1.1

$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$

Solution 1.2

$$\frac{(3n)!(2n+1)!}{(3n+2)!(2n-1)!} = \frac{(3n)!}{(3n+2)!} \times \frac{(2n+1)!}{(2n-1)!} = \frac{1}{(3n+1)(3n+2)} \times 2n(2n+1) = \frac{2n(2n+1)}{(3n+1)(3n+2)}$$

Solution 1.3

$\Gamma(5) = 4! = 24$

Solution 1.4

$\Gamma(4.5) = 3.5\Gamma(3.5) = \cdots = 3.5 \times 2.5 \times 1.5 \times 0.5\Gamma(0.5) = 6.5625\sqrt{\pi} = 11.63$

Solution 1.5

$$\frac{\Gamma(n-1)(2n-1)!}{\Gamma(2n)(n+1)!} = \frac{(n-2)!(2n-1)!}{(2n-1)!(n+1)!} = \frac{1}{n(n+1)(n-1)} \text{ or } \frac{1}{n^3 - n}$$

Solution 1.6

We can prove this using mathematical induction.

When $n=0$, the equation says $\Gamma(½)=\dfrac{0!}{2^0 0!}\sqrt{\pi}=\sqrt{\pi}$, which is correct.

If we assume it's true for a typical value of n, say $n=k$, then we know that:

$$\Gamma(k+½)=\frac{(2k)!}{2^{2k}k!}\sqrt{\pi}$$

This would imply that:

$$\Gamma(k+1½)=(k+½)\Gamma(k+½)=(k+½)\times\frac{(2k)!}{2^{2k}k!}\sqrt{\pi}$$

The last expression can be written as:

$$(k+½)\times\frac{(2k)!}{2^{2k}k!}\sqrt{\pi}=\frac{2k+1}{2}\times\frac{(2k)!}{2^{2k}k!}\sqrt{\pi}=\frac{(2k+1)!}{2^{2k+1}k!}\sqrt{\pi}$$

This doesn't quite match the formula we were hoping for. But, if we include an extra factor of $\dfrac{2k+2}{2(k+1)}$ (which won't affect the answer), we get:

$$\frac{2k+2}{2(k+1)}\times\frac{(2k+1)!}{2^{2k+1}k!}\sqrt{\pi}=\frac{(2k+2)!}{2^{2k+2}(k+1)!}\sqrt{\pi}$$

This now matches the formula given when $n=k+1$. So, if it is true for $n=k$, it's also true for $n=k+1$, and by the principle of mathematical induction it must be true for all values of n.

